The size-effect as a rank-effect

Applications of Stochastic Analysis

Tim Gebbie†‡, Diane Wilcox†, & Raphael Nkomo‡

tim.gebbie@physics.org

School of Computational and Applied Mathematics, University of Witwatersrand†

&

QT Capital Management (Pty) Ltd ‡

Overview

1. Stochastic Portfolio Theory (*work with D Wilcox*)
2. Factorization of Equity Returns (*work with D Wilcox*)
3. Universal Portfolios (*work with R Nkomo*)
4. Universal Anti-correlation Algorithm (*work with R Nkomo*)
5. Appendices

Stochastic Portfolio Theory

Balanced and passive portfolio's ...

Consider a portfolio π represented by weight $\pi_i(t)$ and value $Z_\pi(t)$ at time t

$$Z_\pi = \pi_1 Z_\pi + \ldots + \pi_n Z_\pi,$$

Here $\sum \pi_i = 1$ and π_i is the proportion of Z_π which is invested in X_i, i.e. $\pi_i Z_\pi$ is the amount invested in X_i and $\pi_i Z_\pi = \nu_i X_i$, where ν_i is the number of shares of X_i held s.t.

$$Z_\pi = \nu_1 X_1 + \ldots + \nu_n X_n.$$

- A *balanced portfolio* (BP) is portfolio π such that all the π_i are constant.
- A *passive portfolio* is portfolio such that no. of shares ν_i of each stock X_i is constant.

Market portfolio ...

The *market portfolio* is the passive portfolio μ such that

$$Z_\mu(t) = X_1(t) + \ldots + X_n(t).$$

and the weights $\mu_1, \ldots, \mu_n$ are given by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \ldots + X_n(t)},$$

for all $t \in [0, T]$. The weights $\mu_i(t)$ are called the market weights.

The value of the market portfolio $Z_\mu(t)$ is the combined capitalisation of all the stocks in the market at a given time.

Growth rates of portfolio's ...

For portfolio π

$$dZ_\pi = \sum dX_i = \sum \pi_i Z_\pi \frac{dX_i}{X_i} \quad \Longleftrightarrow \quad \frac{dZ_\pi}{Z_\pi} = \sum \pi_i \frac{dX_i}{X_i}.$$

It follows that:

$$d \log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i,\nu} \pi_i(t)\xi_{i\nu} dW_\nu(t),$$

to find the *portfolio growth rate* $\gamma_\pi(t)$

$$\boxed{\gamma_\pi(t) = \sum_i \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_i \pi_i(t)\sigma_{ii}(t) - \sum_{i,j} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right) \qquad (1)}$$

and volatility

$$\sigma_{ij}(t) = \sum_\nu \xi_{i\nu}(t)\xi_{j\nu}(t)dt$$

.

Excess growth rates & relative returns

Equivalently, for portfolio π with value Z_π we have that

$$d\log Z_\pi(t) = \sum_i \pi_i(t)\gamma_i(t)dt + \gamma_\pi^*(t)dt + \sum_{i,\nu} \pi_i(t)\xi_{i\nu}dW_\nu(t),$$

where the excess portfolio growth rate $\gamma_\pi^*(t)$ is given by

$$\gamma_\pi^*(t) = \frac{1}{2}\left(\sum_i \pi_i(t)\sigma_{ii}(t) - \sum_{i,j} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right). \quad (2)$$

The portfolio relative return of portfolio π with respect to portfolio μ is given by

$$d\log(Z_\pi(t)/Z_\mu(t)) = \sum_i \pi_i(t)d\log\mu_i(t) + \gamma_\pi^*(t)dt. \quad (3)$$

Portfolio generating functions...

Definition *(Relative Portfolio Returns)* [5] Let $\mathbf{S}$ be a positive continuous function defined on Δ^n, and let π be a portfolio. Then $\mathbf{S}$ generates π if there exists a measurable process of bounded variation Θ such that

$$\log(Z_\pi/Z_\mu) = \log \mathbf{S}(\mu(t)) + \Theta(t), \tag{4}$$

for all $t \in [0, T]$, a.s.. The process Θ is called the drift process corresponding to S. If S generates π then S is called the generating function of π, and π is said to be functionally generated.

We can express the portfolio relative returns from Eqn 4 in differential form:

$$d\log(Z_\pi(t)/Z_\mu(t)) = d\log \mathrm{S}(\mu(t)) + d\Theta(t), \quad \text{a.s.,} \quad \text{for all } t \in [0, T]. \tag{5}$$

It is this differential form that is typically worked with when using stochastic portfolio theory methods.

Rank generated portfolio's ...

Theorem 0.1 (Rank Generated Portfolios) *[5] Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ that are pathwise mutually non-degenerate, let p_t be a random permutation of $\{1, \ldots, n\}$, such that for $k = 1, \ldots, n$, the k-th ranked market weight at time t is*

$$\mu_{p_t(k)}(t), \text{and } p_t(k) < p_t(k+1) \text{ if } \mu_{p_t(k)}(t) = \mu_{p_t(k+1)}(t).$$

Let the relative rank covariance process be $\tau_{p_t(ij)} = \tau_{p_t(i)p_t(j)}(t)$ for $i, j = 1, \ldots n$ where the ranked market weights $\mu_{p_t(1)} \geq \ldots \geq \mu_{p_t(n)}$ give $\mu_{p_t} = (\mu_{p_t(1)}, \ldots, \mu_{p_t(n)})$. Let $\mathbf{S}$ be a function defined on a neighborhood U of Δ_n. Suppose that there exists a positive C^2 function S defined on U such that for $(x_1, \ldots, x_n) \in U$, $\mathbf{S}(x_1, \ldots, x_n) = S(x_{p_t(1)}, \ldots, x_{p_t(n)})$, and for $i = 1, \ldots, n$, and $x_i D_i \log S(x)$ bounded for $x \in \Delta^n$. Then $\mathbf{S}(x)$ generates the portfolio π such that for $k = 1, \ldots, n$,

$$\pi_{p_t(k)}(t) = \left(D_k \log S(\mu_{p_t}(t)) + 1 - \sum_{j=1}^{n} \mu_{p_t(j)} D_j \log S(\mu_{p_t}(t)) \right) \mu_{p_t(k)}(t),$$

for all $t \in [0, T]$, a.s., with a drift process Θ.

Rank generated drift process...

The drift process Θ is given by

$$d\Theta(t) = \theta_\pi(t)dt + dL_\pi(t),$$

for all $t \in [0, T]$, a.s. where, a.s., a contribution in terms of time increments

$$\theta_\pi(t) = \frac{-1}{2(S(\mu(t)))} \sum_{i,j=1}^{n} D_{ij} S(\mu_{p_t}(t)) \mu_{p_t(i)}(t) \mu_{p_t(j)}(t) \tau_{p_t(ij)}(t) dt$$

for $t \in [0, T]$, and a contribution in terms of increments in local-times

$$dL_\pi(t) = \frac{1}{2} \sum_{k=1}^{n-1} \left(\pi_{p_t(k+1)}(t) - \pi_{p_t(k)}(t) \right) d\Lambda_{\log \mu_{p_t(k)} - \log \mu_{p_t(k+1)}}(t)$$

and for $t \in [0, T]$.

Discretisation of relative returns...

The ranked portfolio relative returns processes in terms of S and Θ are of interest:

$$d\log(Z_\pi(t)/Z_\mu(t)) = d\log \mathrm{S}(\mu(t)) + d\Theta(t)$$

The distributional component for ranked sub-portfolios can be approximated by

$$d\log(S(\mu(t_0))) \approx \log\left(\frac{S(\mu_{p_{t_1}(1)}(t_1),\ldots,\mu_{p_{t_1}(m)}(t_1))}{S(\mu_{p_{t_0}(1)}(t_0),\ldots,\mu_{p_{t_0}(m)}(t_0))}\right). \quad (6)$$

The drift term includes a smooth contribution and the local-time contribution. The local-time contribution can be approximated as

$$dL_\pi(t_0) \approx \log\left(\frac{S(\mu_{p_{t_0}(1)}(t_1),\ldots,\mu_{p_{t_0}(n)}(t_1))}{S(\mu_{p_{t_1}(1)}(t_1),\ldots,\mu_{p_{t_1}(n)}(t_1))}\right), \quad (7)$$

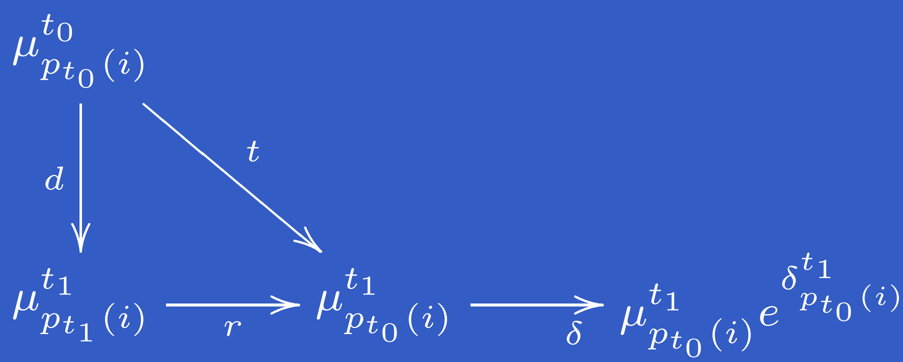
when the time increments from time t to time $t+1$ are sufficiently small. This is the rank component in the factorisation of returns in discrete time.

Factorisation of Equity Returns

Factorisation of returns

Distribution component	$\mu^{t_0}_{p_{t_0}(i)} \longrightarrow_d \mu^{t_1}_{p_{t_1}(i)}$
Rank components	$\mu^{t_1}_{p_{t_1}(i)} \longrightarrow_r \mu^{t_1}_{p_{t_0}(i)}$
Time component	$\mu^{t_0}_{p_{t_0}(i)} \longrightarrow_t \mu^{t_1}_{p_{t_0}(i)}$
Dividend component	$\mu^{t_1}_{p_{t_0}(i)} \longrightarrow_\delta \mu^{t_1}_{p_{t_0}(i)} e^{\delta^{t_1}_{p_t(i)}}$

This can be graphically represented [4] by:



Factorisation of relative returns

The total return of the i-th stock can be factorized from time t_0 to time t_1 :

$$\ln\left(\frac{\mu^{t_1}_{p_{t_0}(i)}}{\mu^{t_0}_{p_{t_0}(i)}}\right) = \ln\left(\frac{\mu^{t_1}_{p_{t_1}(i)}}{\mu^{t_0}_{p_{t_0}(i)}}\right) + \ln\left(\frac{\mu^{t_1}_{p_{t_0}(i)}}{\mu^{t_1}_{p_{t_1}(i)}}\right) + (\delta^{t_1} + \delta^{t_1}_{p_{t_0}(i)}) \quad (8)$$

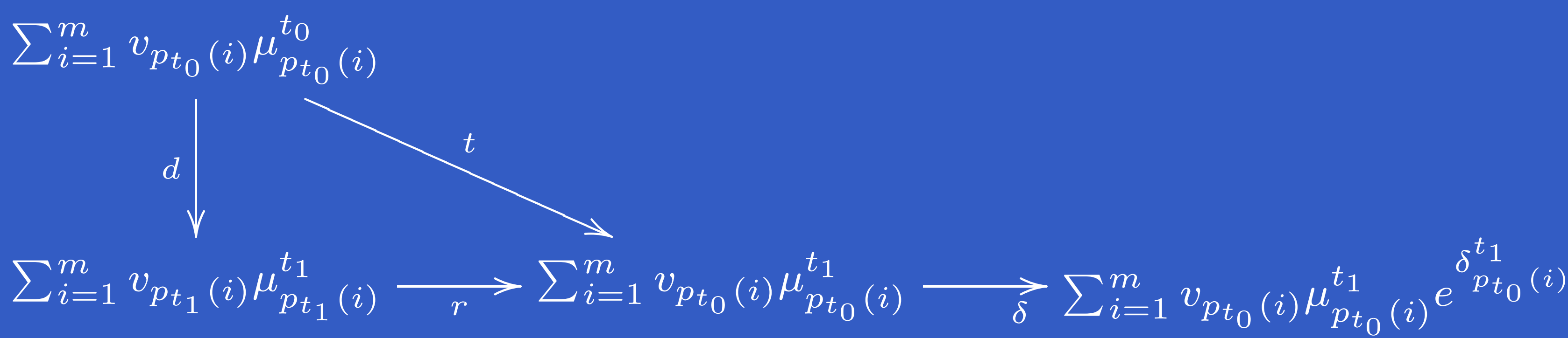
The first term on the right is the distribution contribution, the second is the rank contribution (analogous to the drift contribution in continuous time) and the last contribution is due to the relative contribution of dividends.

Relative to the Market ...

Portfolio π factorized relative to the market portfolio:

$$\ln\left(\frac{V^{t_1}}{V^{t_0}}\right) - \ln\left(\frac{M^{t_1}}{M^{t_0}}\right) = \ln\left(\frac{\mathbf{S}(\mu^{t_1})}{\mathbf{S}(\mu^{t_0})}\right) + \ln\left(\mathbf{R}(\mu^{t_1})\right) + (\delta_v^{t_1} - \delta^{t_1}). \quad (9)$$

Here the portfolio value is V, the generating function of portfolio weights is S and the rank sensitivity function of the portfolio weights is R.
This can be graphically represented [4] by:



Small vs. Large ...

Portfolio	Total	Rank	Distributional	Dividend
m=10,n=250				
Small/Large	4.3 (16.4)	4.6 (6.6)	-0.1 (14.7)	-0.2 (0.3)

The return are annualised monthly returns as %.

1. The size effect can be attributed to *cross-overs* expressed in terms of local-times (rank components) on the JSE (cf.[4, 5] for NYSE results)
2. The theory is descriptive and not normative. It does not provide an explanation of the mechanism causing the cross-overs, but describes how random cross-overs can generate the size effect at the portfolio level.

Exploiting the rank and distributional effects using Universal Portfolios

Universal Portfolios

Definition *(Universal Portfolios)* [2] Price relatives for n-stocks $x(k) = X(k)/X(k-1)$ for times $k = 0 \ldots t$ form a t-tuple as $x^t = x_1^t = x(1), \ldots, x(t)$. Let μ be a prior distribution of portfolios on simplex Π. An Expert is a given realization π from this distribution of portfolios. The μ-weighted universal portfolio at time t is:

$$\hat{\pi}(t) = \frac{\int_\Pi \pi Z_\pi(t-1) d\mu(\pi)}{\int_\Pi Z_\pi(t-1) d\mu(\pi)}, \quad (10)$$

with $\int_\Pi d\mu(\pi) = 1$, where the portfolio value Z_π at time t is:

$$Z_\pi(t, x^t) = Z_\pi(t) = \prod_{k=1}^{t} \pi \cdot x(k) = \prod_{k=1}^{t} \sum_{i=1}^{n} \pi_i x_i(k), \quad \text{and} \quad Z_\pi(0) = 1. \quad (11)$$

The relative performance to the best BP is: $\frac{Z_{\hat{\pi}}(t)}{Z_{\max_\pi\{Z_\pi(t)\}}} \geq \frac{1}{(t+1)^{n-1}}$.

Universal Anti-correlation Algorithm

Definition *(Anti-correlation Algorithm)* For each stock i and j we compute the $\mathsf{claim}_{i\to j}$, this is the extent to which capital is to be shifted from stock i to stock j based on cross-correlations $M_{cor}(i,j)$ and return differentials μ_2 in consecutive partially overlapping windows of size ω. If $\mu_2(i) > \mu_2(j)$ and $M_{cor}(i,j) > 0$ there is a claim:

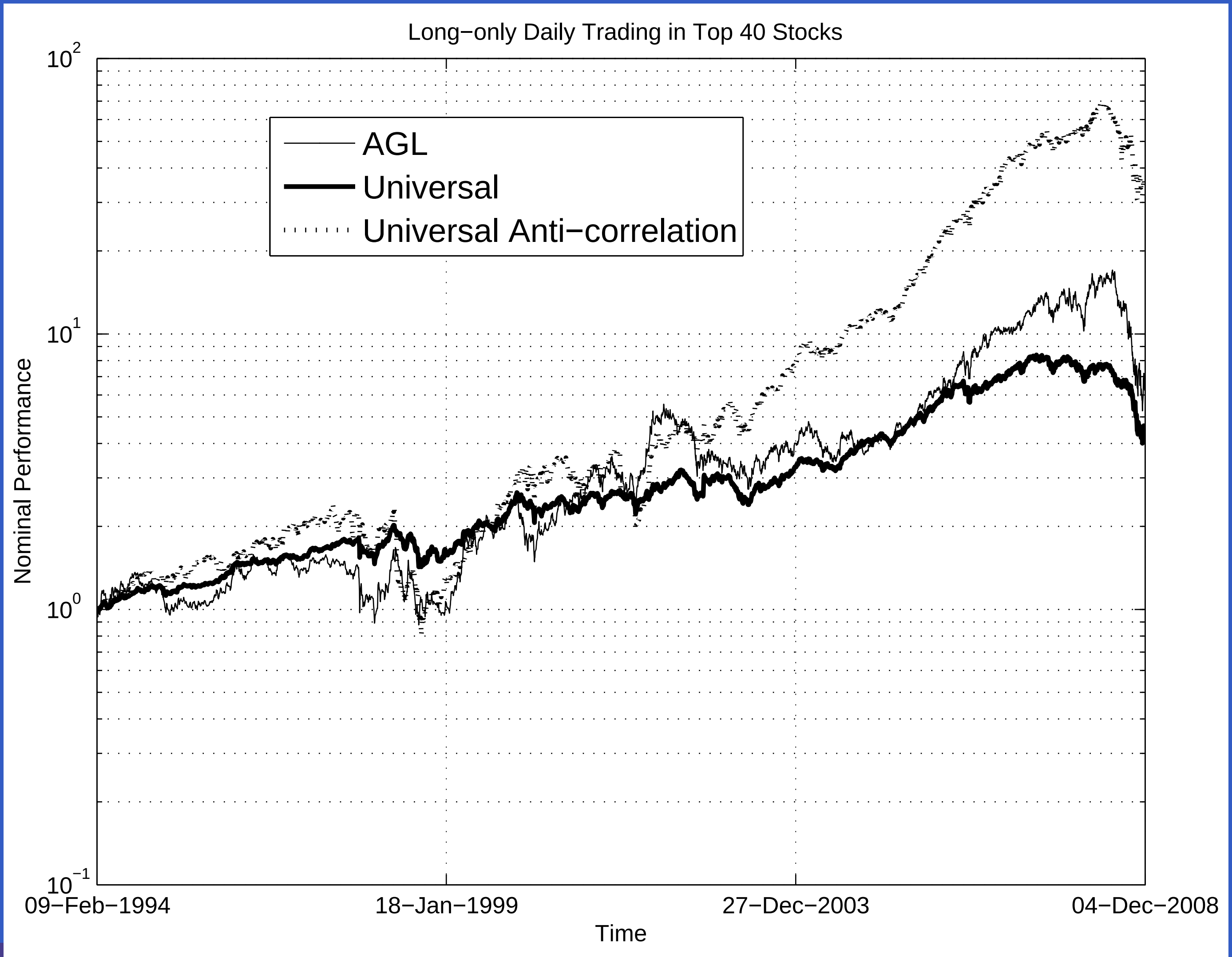
$$\mathsf{claim}_{i\to j} = M_{cor}(i,j) + A(i) + A(j), \tag{12}$$

where $A(h) = |M_{cor}(h,h)|$, if, $M_{cor}(h,h) < 0$, else 0. From the claims the weights are computed using the expected transfers of capital:

$$\pi_i^\omega(t+1) = \pi_i^\omega(t) + \sum_{j\neq i} \left[\mathsf{transfer}_{j\to i} - \mathsf{transfer}_{i\to j}\right], \tag{13}$$

where $\mathsf{transfer}_{i\to j} = \pi_i^\omega(t)\mathsf{claim}_{i\to j}/\sum_j \mathsf{claim}_{i\to j}$. The transfers are computed based on the extent to which stocks are correlated with each other in consecutive partially overlapping windows of size ω. An expert $\pi^\omega(t)$ is associated with each choice of window size ω.

Universal Anti-correlation Algorithm



The End

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[2] Cover,T.,Ordentlich,E., (1996) Universal Portfolios with side information, *IEEE Transactions on Information Theory*, 42(2), 348-363

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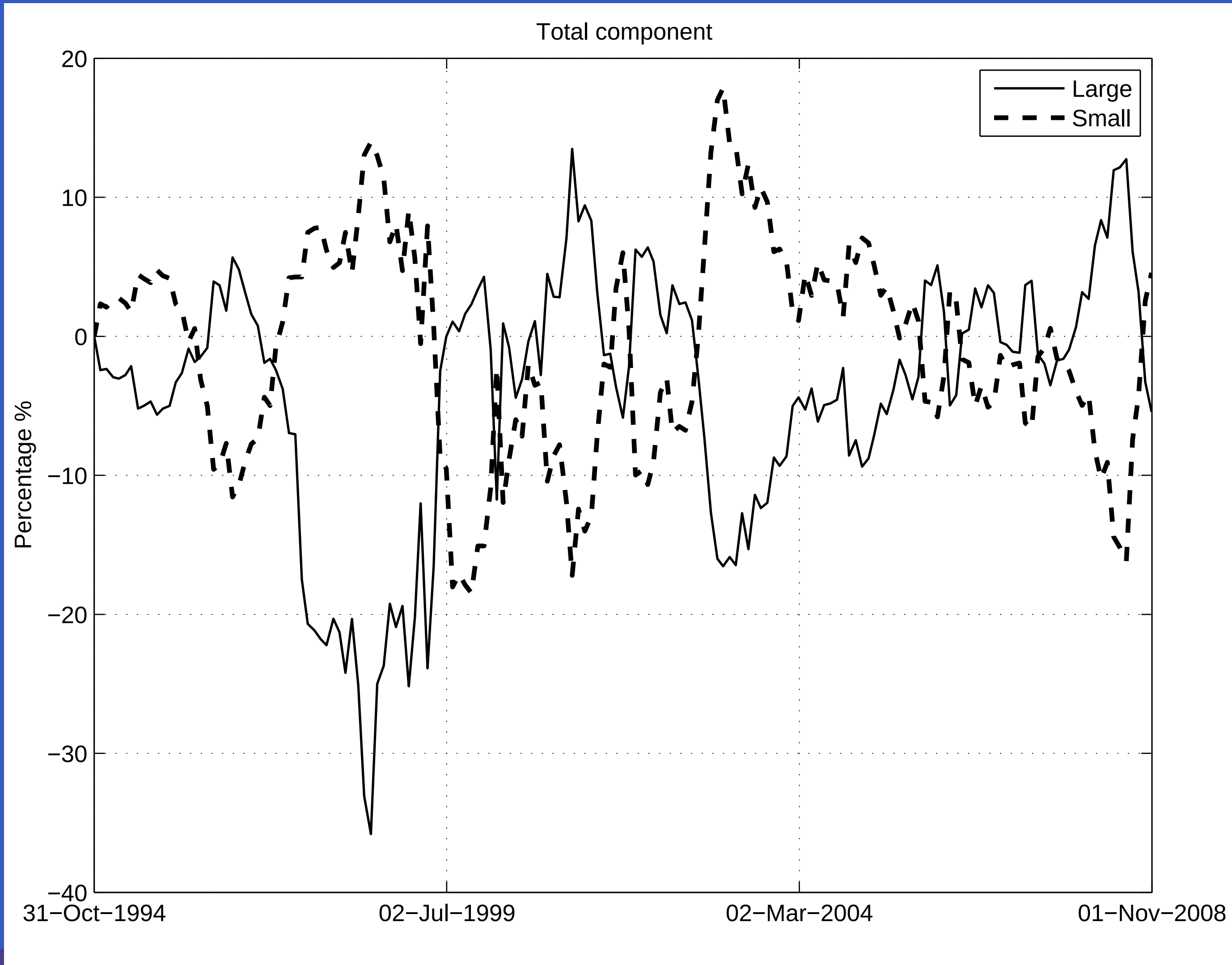
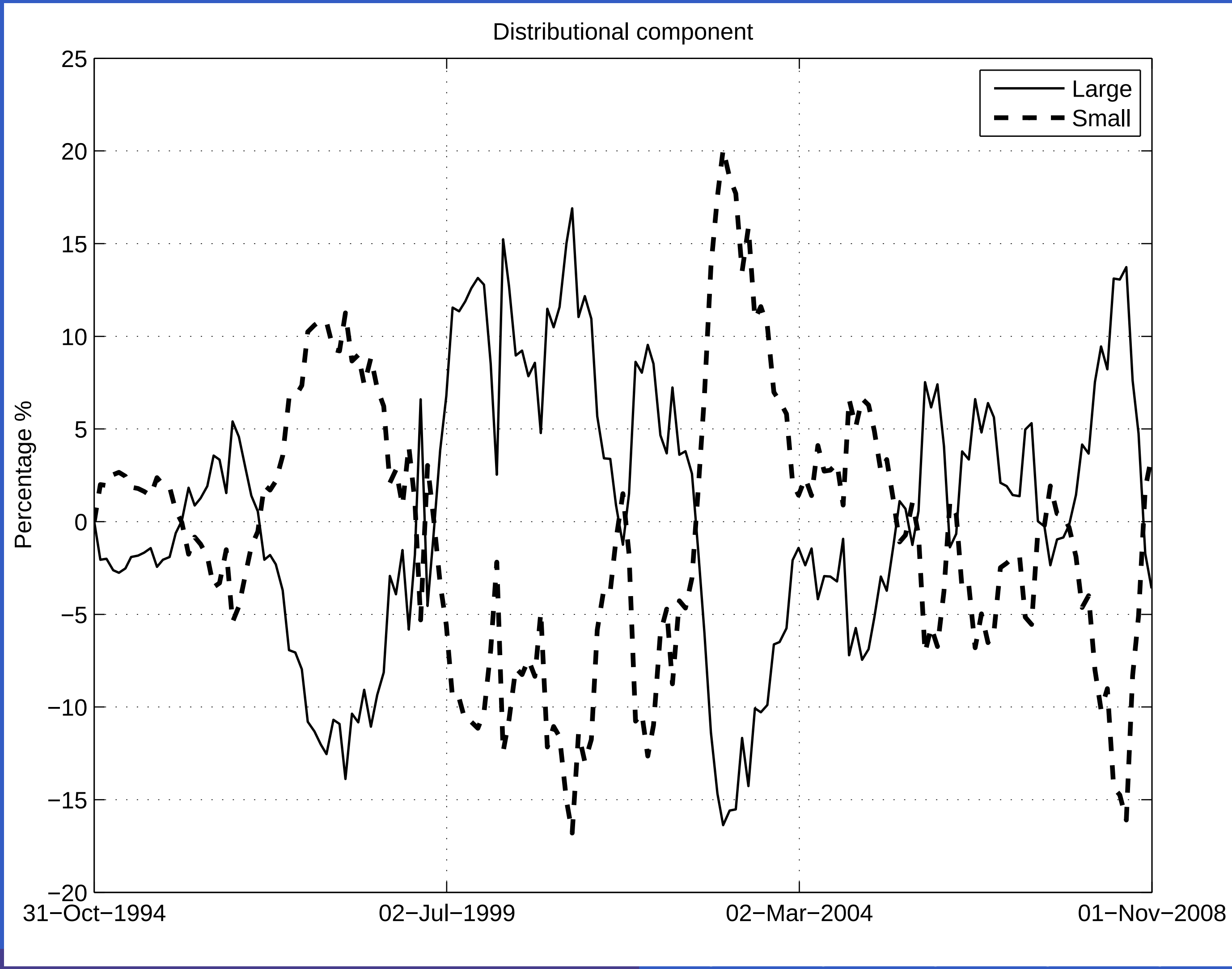
[5] Fernholz, R., Karatzas, I., (2006) The implied liquidity premium for equities, *Annals of Finance*, (2), 87-99.

[6] Wilcox, D., Gebbie, T., (2007) The Factorisation of equity returns in South African markets, Working Paper (in preparation)

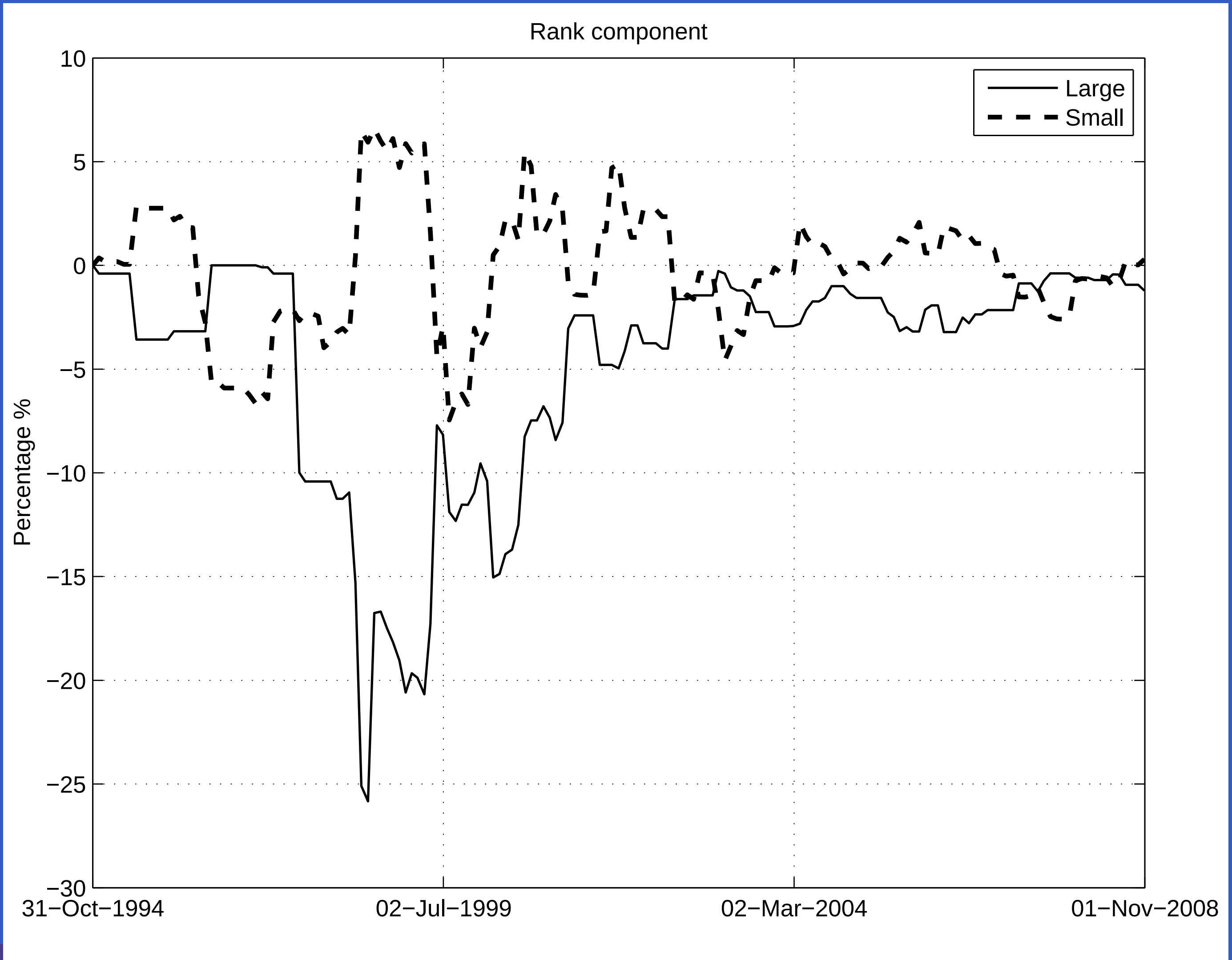
Appendix A: Total



Appendix A: Distributional



Appendix A: Rank



Appendix A: Dividend